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OPTIMAL COMMODITY PROCUREMENT UNDER STOCHASTIC PRICES

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ABSTRACT

This paper presents a forward buying optimization problem central to many commodity procurement decisions. Specifically, we present a new mode of analysis to inventory management systems when the unit-purchasing cost is allowed to be a general stochastic process. Assuming dynamic but deterministic demand, negligible fixed ordering cost, and constant lead time, we derive recursive expressions to compute the optimal number of periods of demand to purchase for a given price model. Our problem can also be described as the optimal forward buying quantity for a given price model. We show that the computation time of the optimal policy exponentially increases in the number of future periods. Therefore, an heuristic policy is developed based on computationally efficient bounds of the optimal cost function. Since the procurement model is price model free, it is easy to implement for a variety of commodities with different price processes.

KeyWords: Forward Buying; Case Study; Optimization; Stochastic Prices; Inventory Management; Procurement

RASTSAL FIYAT ORTAMINDA OPTİMAL SATINALMA

ÖZET

Bu makale işletmelerin satınalma kararlarında esas olan ileriye dönük satınalmanın optimizasyonu problemini sunmaktadır. Diğer bir deyişle, birim alış maliyetinin genel bir olasılık fonksiyonu olduğu envanter yönetim sistemlerine yeni bir analitik yaklaşım sunuyoruz. Dinamik fakat deterministik bir talep fonksiyonu, gözardı edilebilir sabit sipariş maliyeti ve sabit sipariş alım süresini kabul ederek, verilen herhangi bir fiyat modeli için optimal gelecek süre için mal siparişini hesaplayan formüller türetiyoruz. Problem, verilen herhangi bir fiyat modelinde gelecek için sipariş edilmesi gereken optimal stok miktarı olarak ta tanımlanabilir. Optimal politikanın hesaplanması için gereken sürenin gelecek zaman dilimlerinin sayıları üzerinde üstsel olarak arttığı (hesaplamanın zorlaştığı) görülür. Bu sebeple, optimal maliyet fonksiyonunun alt ve üst sınırları geliştirilerek yaklaşık alt-üst sipariş politikaları türetilmiştir. Geliştirilen sipariş politikası fiyat modelinden bağımsız olduğundan değişik fiyat modellerine sahip mallar ve hammadde alımlarında uygulanması kolaydır.

Anahtar Kelimeler: İleriye Dönük Satınalma, Optimizasyon, Rastsal Fiyatlar, Envanter Yönetimi, Satınalma

1 INTRODUCTION

This research focuses on a forward buying decision that can be classified as an inventory procurement problem under stochastic prices. It was motivated by an application involving the procurement of raw material for a firm that produces manufactured (or mobile) homes. The firm needed a systematic means of trading off critical cost and risk factors, such as the dynamics of future unit prices, inventory costs, and projections of upcoming production requirements. The work report here was the center piece of that application – the model used to incorporate stochastic prices into a periodic review replenishment policy.

There has been much work on inventory related models since the 1950's, but little of it allows stochastic prices. This research develops an optimal commodity procurement model, along with its computationally efficient bounds. Specifically, this study considers inventory procurement strategies based on explicit consideration of stochastic price process which may be stationary or non-stationary. Since we are not considering any specific price process, we will not attempt to express the expected inventory cost function in closed form. In such cases, Monte Carlo simulation can be used to estimate the minimum expected discounted costs; however this tactic is computationally intensive. Therefore, some computationally efficient bounds are developed.

Fabian et al. (1959) study a multiple period inventory procurement problem under probabilistic prices and probabilistic demand, with no discounting. Although they are able to show the optimality of price-dependent base stock policy for stationary prices, they fail to generalize the optimality of this policy to nonstationary case (Part I of their work). Kingsman (1969) considers a finite horizon commodity-purchasing problem under stochastic market prices so as to minimize long-term costs where the buyer has many opportunities to make a purchase. He suggested price-dependent base stock policy, which was later proved by Golabi (1985) for the stationary and nonstationary price distribution cases. Kalymon (1970) extends Scarf's (1960) inventory model to include stochastic prices. His work is more general in that fixed costs and random demand as well as stationary/non-stationary Markovian prices are allowed, and it can be shown that his price dependent Reorder-Point Order-Up-to policy is a generalization of price dependent base stock policy of the above authors. Kalymon, however, does not provide a practical computable policy other than a mere application of dynamic programming. Shastry (1993) studied the procurement problem of manufactured goods whose prices change in discrete jumps. Timing between two successive price jumps is a random variable with a known probability density function. His problem is about manufactured goods; therefore, it does not exactly fit in the commodity procurement context.

Our research is significantly different from the previous work in several ways: First, we do not restrict prices to a specific form. Second, the developed algorithm is not dependent upon the specific assumptions of the nature of the price distribution, which results in low maintenance as the model for prices changes over time and across items. Third, it is computationally efficient due to the use of the developed bounds and policies. Finally, we present a new mode of analysis of the inventory procurement problem. The rest of the paper is as follows: in §2 single item inventory procurement problem is defined and the optimal policy is developed. In §3 lower and upper bounds on the optimal cost function and their respective policies are developed. This paper concludes in §4.

2. SINGLE ITEM PROCUREMENT

We consider a single item, infinite horizon, stochastic price and deterministic demand commodity procurement problem. The ordering price per unit follows a nonnegative discrete time stochastic process. Constant lead-time, constant discount rate, and no fixed ordering cost are assumed. No backlogging is allowed. The procurement decision is made at discrete points in time and the procurement cost is incurred when an order is placed. The sequence of events for inventory control at each period is given below:

1. At the beginning of the period:
2. Price quote is obtained,
3. Desired inventory position is calculated,
4. Current inventory position is observed, and
5. If current inventory position is less than desired, an order is placed and procurement costs are incurred.
6. At the end of the period:
7. Demand is observed, and
8. Inventory holding cost is charged.

2.1. PROCUREMENT MODEL

The following notation is used in developing the optimization logic and model used to guide the decision maker in executing the process outlined above:

t = Period index

z_t = Realized price in period t
$z_{t,t+i}$ = Forecast made in period t for period t+i
u_t = Forecast error in period t based on forecast made in period t-1
D_t = Demand in period t
h = Holding cost per unit per period
IP_t = Inventory position in period t
L = Lead time
Q_t = Quantity ordered in period t
α = Discount rate
H_n = Cumulative discounted carrying costs, defined as:

$$H_n = \alpha^L h + \alpha^{L+1} h + \cdots + \alpha^{L+n-1} h = \alpha^L h \frac{1-\alpha^n}{1-\alpha}$$

$E_t[\cdot]$ = Expectation conditional on information known at time t

Figure 1 shows that two possible purchase points remain for period t+L+1, namely periods t (the current period) and t+1. Since demand is assumed to be known, current inventory should be enough to meet demand over periods t through t+L-1 and period t+L's demand must be purchased in period t if inventory is not sufficient to meet that period's demand. Period t+L+1 is the first period for which purchasing its demand is optional.

There are three possible sources to meet demand in period t+L+1: current inventory in period t, inventory ordered in period t, and inventory ordered in period t+1. The decision is straightforward in this case. Meet demand in period t+L+1 from inventory if current inventory is sufficient, because this would be the least cost option (no procurement is made and inventory already on-hand is held for a shorter period resulting in lower inventory holding cost). If current inventory is not sufficient to meet demand for period $t+L+1$, then we must purchase its demand in periods t or t+1. If we purchase in period t (now): each unit purchased will incur the procurement cost z_t, the order will arrive L periods later (at the beginning of period t+L), and a one period unit carrying cost (H_1) will be incurred for each unit resulting in a cost of $z_t + H_1$ per unit. If we wait until the next period, t+1, we must purchase at that time and incur a unit procurement cost of z_{t+1}. Since this cost is incurred a period later, it is discounted to be comparable to the period t purchase cost.

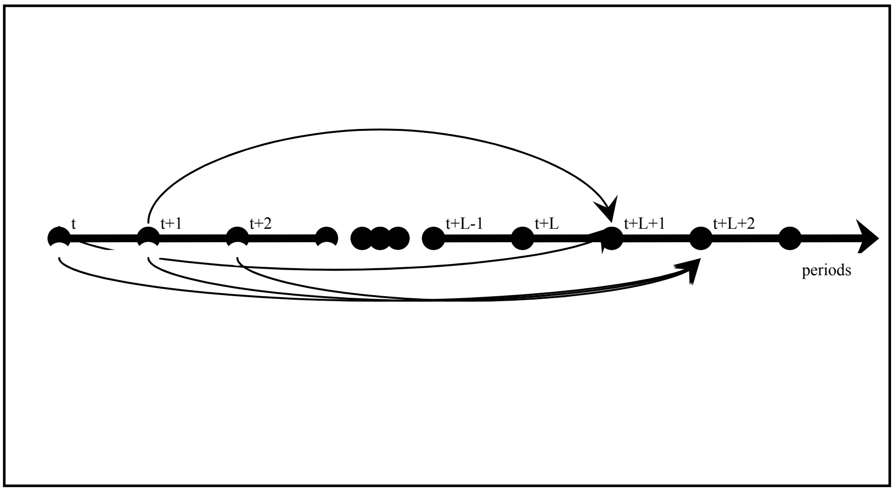

Figure 1. Schema for meeting demand in periods t+L+1 and t+L+2

The order will arrive in period t+L+1, so there will be no carrying cost. Given that current inventory is not sufficient to meet demand through period t+L+1 and backlogging is not allowed, we must decide whether to order now or wait until the next period so that the total expected cost of meeting demand in period t+L+1 is minimized. We minimize these costs by ordering now if the current unit procurement cost, plus unit-carrying cost, is less than the discounted expected unit cost next period. That is, it is optimal to purchase in period t for period t+L+1 if

$$z_t + H_1 < \alpha E_t[z_{t+1}] = \alpha E_t[z_{t,t+1} + u_{t+1}] = \alpha z_{t,t+1}$$

otherwise, wait.

The decision for period t+L+2 is more involved; we can buy in period t or wait. If we buy in period t for period t+L+2, the relevant costs are: unit procurement cost, z_t, and discounted unit carrying cost over periods t+L and t+L+1, H_2. Therefore, total cost per unit of meeting demand in period t+L+2 is: $z_t + H_2$. If we wait, we have two options in period t+1 to meet demand in period t+L+2: (1) Purchase in period t+1 and incur a cost of $z_{t+1} + H_1$ (where H_1 is the carrying cost over period t+L+1). (2) Do not purchase in period t+1 and incur an expected discounted cost per unit of $\alpha E_{t+1}[z_{t+2}]$. Therefore, as of period t+1, the optimal decision is to select the minimum of (1) and (2). Thus, the minimum expected cost of buying in period t+1 is $\min(z_{t+1} + H_1, \alpha E_{t+1}[z_{t+2}])$. Since both terms inside the min function are unknown in period t, taking the expectation and discounting back to period t, the expected discounted cost of waiting (not buying in period t) is:

$$\alpha E_t[\min(z_{t+1} + H_1, \alpha E_{t+1}[z_{t+2}])].$$

Therefore, it is optimal to buy in period t for period $t + L + 2$ if

$$z_t + H_2 < \alpha E_t[\min(z_{t+1} + H_1, \alpha E_{t+1}[z_{t+2}])]$$

otherwise, wait.

In general, the decision can be visualized as comparing the cost of buying now versus a nested expectation of minimums of discounted future costs. For example, the optimal decision is to buy in period t for period t+L+3 if

$$z_t + H_3 < \alpha E_t[\min(z_{t+1} + H_2, \alpha E_{t+1}[\min(z_{t+2} + H_1, \alpha E_{t+2}[z_{t+3}])])],$$

otherwise, wait.

To generalize, let us define

$$R_{t,1} = \alpha E_t[z_{t+1}] = \alpha z_{t,t+1}$$

$$R_{t,2} = \alpha E_t[\min(z_{t+1} + H_1, \alpha E_{t+1}[z_{t+2}])]$$

$$= \alpha E_t[\min(z_{t+1} + H_1, R_{t+1,1})] \quad \text{and in general,}$$

$$R_{t,n} = \alpha E_t[\min(z_{t+1} + H_{n-1}, R_{t+1,n-1})].$$

Then, the optimal decision is to buy in period t for period $t + L + n$ if the expected savings

$$S_{t,n} = R_{t,n} - z_t - H_n$$

is positive. We now show that $S_{t,n}$ is a non-increasing sequence in n. This property implies that if it is optimal to buy now for a future period k, it is also optimal to buy now for all the periods before k. It also implies that if it is optimal not to buy for a future period k, it is also optimal not to buy for any period beyond k. Therefore, periods should be added to the current buy until the expected savings turns negative.

Optimal Policy: In order to calculate the optimal order quantity, it is useful to first state the following proposition.

Proposition 1: $S_{t,n} \geq S_{t,n+1}$ for all $t, n \geq 1$. The proof for this proposition is included in the Appendix (as are the proofs for all subsequent propositions).

Now we are ready to define the optimal ordering policy as follows:

If $S_{t,1} \leq 0$, then the optimal forward buying critical number, k_t, is zero. Otherwise, there exists a unique integer k_t such that $S_{t,k_t} > 0$ and $S_{t,k_t+1} \leq 0$. Then, the optimal order quantity is

$$Q_t = \max\left\{ \sum_{i=t}^{t+L+k_t} D_i - IP_t \, , 0 \right\}$$

where D_i is known demand in period i and IP_t is on-hand plus on-order inventory.
If $S_{t,1} \leq 0$, then by Proposition 1, $S_{t,n} \leq 0$ for all n, which means that there is no incentive to buy now for the demand in period t+L+n and hence no forward buying takes place. On the other hand, if $S_{t,1} > 0$ then by Propositions 1 and 8 (see section 3.1), there exists a smallest integer k such that $S_{t,k} > 0$ but $S_{t,k+1} \leq 0$. This means that there are positive expected savings for purchasing now for periods t+L+1 through t+L+k and no savings for purchasing now for periods after t+L+k. In this case, the policy orders in period t for periods through t+L+k if the current inventory position is not sufficient to meet demand in these periods. If the current inventory is more than demand for periods t through t+L+k, then the policy orders nothing. If the assumption of Proposition 8 does not apply, it may be the case that $S_{t,k} > 0$ for all t and k. There would then be an incentive to order now for every future period.

It remains to be shown how to calculate $R_{t,n}$. Since $R_{t,n}$ involves the expectations of correlated random variables (z_t can be any stochastic process including an ARIMA process where z_t depends on its own lags and past disturbances), it is not possible to obtain a closed form expression for $R_{t,n}$. One approach to computing $R_{t,n}$ is to use brute force Monte Carlo (MC) estimation. For large n, brute force MC turns out to be very ineffective. Computation time grows exponentially as n is increased. If an average is taken over m price realizations in any period, then m z_{t+1} prices are needed to estimate $R_{t,1}$, m^2 z_{t+2} prices to estimate $R_{t,2}$, and in general, m^n z_{t+n} prices are needed to estimate $R_{t,n}$. This corresponds to an exponential computation time. For large n, therefore, brute force MC estimation is inefficient if not infeasible. Therefore, we develop computationally efficient bounds in the next section.

3 BOUNDS

In this section we develop a lower and an upper bound that give minimum and maximum quantities to buy respectively. Let us define

$$L_{t,n} = \alpha E_t\left[\min_{1\leq i\leq n} \alpha^{i-1}(z_{t+i} + H_{n-i}) \right], \quad t \geq 1, n \geq 1 .$$

In other words, if we knew all the prices, $\min_{1\leq i\leq n} \alpha^{i-1}(z_{t+i} + H_{n-i})$ is the future lowest cost for period n. Since we only know prices until period t (now), we take the expectation over all possible price sample paths for periods t+1 through t+n. Since prices and holding costs are nonnegative, $L_{t,n}, t \geq 1, n > 1$ is nonnegative. The following proposition establishes the lower bound.

Proposition 2: $L_{t,n} \leq R_{t,n}, \forall t, \forall n$.
Let us define the minimum expected savings function as:

$$S_{t,n}^{Min} = L_{t,n} - z_t - H_n .$$

Then, $S_{t,n}^{Min}$ provides a lower bound for $S_{t,n}$. The following proposition establishes the monotonicity of $S_{t,n}^{Min}$.

Proposition 3: $S_{t,n}^{Min} \geq S_{t,n+1}^{Min}$ for $n \geq 1$ and all t.
With these propositions, we are in a position to define the lower bound policy as follows:

If $S_{t,1}^{Min} \leq 0$, then the lower bound critical number, $k_t^{Min} = 0$. Otherwise, the k_t^{Min} is the unique integer k such that
$S_{t,k}^{Min} > 0$ and $S_{t,k+1}^{Min} \leq 0$. Then, the lower bound order quantity,

$$Q_t^{Min} = \max\left\{ \sum_{i=t}^{t+L+k_t^{Min}} D_i - IP_t , 0 \right\}$$

where D_i is known demand in period i and IP_t is on-hand plus on-order inventory.
The next proposition shows that the lower bound policy procures no more than the optimal policy.

Proposition 4: $Q_t^{Min} \leq Q_t$.
Proof: $0 < S_{t,k_t^{Min}}^{Min} \leq S_{t,k_t^{Min}} \Rightarrow k_t \geq k_t^{Min}$

Since $S_{t,n}$ is difficult to compute, we have no way to test how close $S_{t,n}^{Min}$ to $S_{t,n}$ is. Therefore, to constrain $S_{t,n}$, an upper bound is needed.

3.1.UPPER BOUND

In this section we develop an upper bound for $S_{t,n}$. Let

$$U_{t,n} = \alpha \min_{1 \leq i \leq n} \left\{ \alpha^{i-1} (E_t[z_{t+i}] + H_{n-i}) \right\} \quad n \geq 1 .$$

If we had perfect forecasts, $U_{t,n}$ would give the optimal periods to cover. Since prices and holding costs are nonnegative, $U_{t,n}$, $t \geq 1$ and $n \geq 1$ is nonnegative.

Proposition 5: $U_{t,n} \geq R_{t,n}$ for $n \geq 1$ and $t \geq 1$.
Let us define the maximum expected savings function, $S_{t,n}^{Max}$, as:
$$S_{t,n}^{Max} = U_{t,n} - z_t - H_n .$$
$S_{t,n}^{Max}$ has the same monotonic property as $S_{t,n}^{Min}$ and $S_{t,n}$.

Proposition 6: $S_{t,n}^{Max} \geq S_{t,n+1}^{Max}$ for $n \geq 1$ and all t.

The next proposition proves that $S_{t,n}^{Max}$ is a monotonically decreasing sequence in n with a finite negative limit. Hence, the sequence can change signs at most once. This limit is obviously true for stochastically decreasing and stationary/covariance stationary prices. To establish stochastically increasing prices, we assume: (1) $0 \leq \alpha < 1$ and (2) the expected future prices increase slower than the discount rate. If these assumptions hold, then the expected discounted future prices approach zero in the limit. If the second assumption does not hold, then the expected prices increase faster than the discount rate and depending on the magnitude of holding cost, it may be optimal to order for infinite number of periods. In fact, without these two assumptions note that the lower bound policy may order for infinite number of periods, as does the optimal policy. Proposition 7 also utilizes the above assumptions 1 and 2.

Proposition 7: Suppose $\lim_{n\to\infty} \alpha^n E[z_{t+n}] = 0 \ \forall t$. Then

$$\lim_{n\to\infty} S_{t,n}^{Max} = -(z_t + H_\infty).$$

The following proposition generalizes this result to $S_{t,n}$ and $S_{t,n}^{Min}$.

Proposition 8: Suppose $\lim_{n\to\infty} \alpha^n E[z_{t+n}] = 0 \ \forall t$. Then

$$\lim_{n\to\infty} S_{t,n} = \lim_{n\to\infty} S_{t,n}^{Min} = -(z_t + H_\infty).$$

Proposition 8 shows that the optimal expected savings, as well as the minimum expected savings, will eventually become negative and approach to the same quantity, $-(z_t + H_\infty) = -(z_t + \frac{\alpha^L}{1-\alpha}h)$, as does the maximum savings function. This rules out the optimality of an order quantity for an infinite number of periods when the above conditions are met.

Using the above propositions we can now define an upper bound policy. Specifically, if $S_{t,1}^{Max} \leq 0$, then the upper bound critical number, $k_t^{Max} = 0$. Otherwise, k_t^{Max} is the unique integer k such that $S_{t,k}^{Max} > 0$ and $S_{t,k+1}^{Max} \leq 0$. Then, the upper bound order quantity,

$$Q_t^{Max} = \max\left\{ \sum_{i=t}^{t+L+k_t^{Max}} D_i - IP_t \, , 0 \right\}$$

where D_i is known demand in period i and IP_t is On-Hand plus On-Order inventory.

Note that if the assumptions of Proposition 7 do not hold, the upper bound policy may order for an infinite number of periods, as does the optimal policy. The next proposition shows that the upper bound policy orders at least as much as the optimal policy.

Proposition 9: $Q_t^{Max} \geq Q_t$.

Proof: $S_{t,k_t^{Max}+1} \leq S_{t,k_t^{Max}+1}^{Max} \leq 0 \Rightarrow k_t \leq k_t^{Max}$

4. CONCLUSION AND FUTURE RESEARCH

This paper has investigated the forward buying problem in a stochastic price environment. Although some simplifying assumptions are made concerning the inventory procurement problem, we have allowed prices to be any general stochastic process. We have developed the optimal policy and its bounds for such an inventory system. In another paper that will be published soon in this journal, we have applied these policies as well as an operational policy that is developed in that paper on the actual prices in a case study and showed that the operational policy outperformed the upper bound and lower bound policies while all of the policies outperformed the myopic policy.

Our model considers any stochastic price process but treats demand as deterministic. Allowing stochastic demand would not change the optimal number of periods to purchase. If it is optimal to buy for a future period under deterministic demand, it is also optimal to buy for that period under stochastic demand. However, when demand is uncertain, it is not known what purchase quantity would exactly meet the desired number of periods of demand. A heuristic would be to substitute expected values of future demands into the optimal order quantity $Q = \max\left\{E\left[\sum_{i=t}^{t+L+k} D_i\right] - IP_t, 0\right\}$.

If demand is larger than what we have bought, then we have missed on savings since the expected cost of not meeting demand over the periods t through t+L+k is higher than purchasing now. If demand is less, then we incur more carrying cost and lose on savings since it is more expensive to buy inventory now and carry for periods beyond period t+L+k.

We conjecture that under stochastic demand (possibly price dependent) with a fixed ordering cost, the optimal policy is a price sample path dependent (s,S) policy. Kalymon's policy, $(s_i(p), S_i(p))$, depends only on the current price p since he assumes prices follow a first order Markovian process. Any ARMA price/demand process can be converted into a Markovian process by state augmentation - including past prices/demands and their respective disturbances into the state vector. See, for example, Bertsekas (1995, pp. 230) and Zipkin (2000, pp. 479). Extending Kalymon's results to general ARMA/ARIMA price models needs further research, although a history dependent (s,S) policy is most probably the optimal policy.

Hence, allowing stochastic demand with/without a positive fixed ordering cost would most likely result in extensions of well known optimal policies of inventory theory. However, even if we know the form of the optimal policy, computing these policies is a nontrivial task. Only recently, computationally efficient algorithms for single pair stationary (s,S) policy have been reported in the literature. Computing nonstationary, let alone sample path dependent nonstationary (s,S) policies is a formidable task since future distributions would change as we compute these critical numbers.

Our model does not allow for shortages either. Though allowing for shortages makes more sense under stochastic demand, it may also be necessary to allow for shortages under the deterministic demand. Depending on the relative magnitude of the cost of shortage compared to the unit purchase cost and cumulative discounted holding costs; it may not always be optimal to meet the known demand when the current price is high. Therefore, allowing backorders/lost sales into our model while demand is still deterministic is an immediate next research interest.

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APPENDIX (PROOFS)

PROOF OF PROPOSITION 1. By induction. Since $S_n = R_{t,n} - z_t - H_n$ and $S_{n+1} = R_{t,n+1} - z_t - H_{n+1}$, first show true for n=1 and all t:

$R_{t,1} - z_t - H_1 \geq R_{t,2} - z_t - H_2$ iff

$R_{t,2} + H_1 - H_2 \leq R_{t,1}$ iff

$\alpha E_t[\min(z_{t+1} + H_1, R_{t+1,1})] - \alpha^{L+1}h \leq \alpha E_t[z_{t+1}]$ iff

$E_t[\min(z_{t+1}, R_{t+1,1} - \alpha^L h)] \leq E_t[z_{t+1}]$ which is obviously true.

Now assume true for n = k and all t and show true for n = k + 1 and all t:

$R_{t,k+1} - z_t - H_{k+1} \geq R_{t,k+2} - z_t - H_{k+2}$ iff

$R_{t,k+2} + H_{k+1} - H_{k+2} \leq R_{t,k+1}$ iff

$\alpha E_t[\min(z_{t+1} + H_{k+1}, R_{t+1,k+1})] - \alpha^{L+k+1}h \leq \alpha E_t[\min(z_{t+1} + H_k, R_{t+1,k})]$ iff

$$E_t[\min(z_{t+1} + H_k, R_{t+1,k+1} - \alpha^{L+k}h)] \leq E_t[\min(z_{t+1} + H_k, R_{t+1,k})] \quad (1)$$

Now by the induction assumption we have

$R_{t+1,k} - z_{t+1} - H_k \geq R_{t+1,k+1} - z_{t+1} - H_{k+1}$ iff

$R_{t+1,k+1} - \alpha^{L+k}h \leq R_{t+1,k}$

So clearly (1) holds and the induction holds.

PROOF OF PROPOSITION 2. By induction. $L_{t,1} \leq R_{t,1}$ since $L_{t,1} = R_{t,1}$.

Assume that $L_{t,n} \leq R_{t,n}\ \forall t$. It remains to show that $L_{t,n+1} \leq R_{t,n+1},\ \forall t$

$$L_{t,n+1} = \alpha E_t\left[\min_{1\leq i\leq n+1} \alpha^{i-1}\left(z_{t+i} + H_{n+1-i}\right)\right]$$

$$= \alpha E_t\left[\min\left(z_{t+1} + H_n, \min_{2\leq i\leq n+1} \alpha^{i-1}\left(z_{t+i} + H_{n+1-i}\right)\right)\right]$$

$$= \alpha E_t\left[E_{t+1}\left[\min\left(z_{t+1} + H_n, \min_{2\leq i\leq n+1} \alpha^{i-1}\left(z_{t+i} + H_{n+1-i}\right)\right)\right]\right]$$

Since the expectation of a minimum is less than or equal to the minimum of the expectations,

$$\leq \alpha E_t[\min(z_{t+1} + H_n, E_{t+1}[\min_{2\leq i\leq n+1} \alpha^{i-1}(z_{t+i} + H_{n+1-i})])]$$

$$= \alpha E_t[\min(z_{t+1} + H_n, \alpha E_{t+1}[\min_{2\leq i\leq n+1} \alpha^{i-2}(z_{t+i} + H_{n+1-i})])]$$

By re-indexing,

$$= \alpha E_t[\min(z_{t+1} + H_n, \alpha E_{t+1}[\min_{1\le i\le n} \alpha^{i-1}(z_{t+1+i} + H_{n-i})])]$$
$$= \alpha E_t[\min(z_{t+1} + H_n, L_{t+1,n})]$$
$$\le \alpha E_t[\min(z_{t+1} + H_n, R_{t+1,n})]$$
$$= R_{t,n}.$$

PROOF OF PROPOSITION 3. $L_{t,n} - z_t - H_n \ge L_{t,n+1} - z_t - H_{n+1}$ if

$$L_{t,n+1} + H_n - H_{n+1} \le L_{t,n}.$$

Now

$$L_{t,n} = \alpha E_t[\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i})]$$

and

$$L_{t,n+1} + H_n - H_{n+1} = \alpha E_t[\min_{1\le i\le n+1} \alpha^{i-1}(z_{t+i} + H_{n+1-i})] - \alpha^{L+n}h$$
$$= \alpha E_t[\min\{\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n+1-i} - \alpha^{L+n-i}h), \alpha^n(z_{t+n+1} - \alpha^{L-1}h)\}]$$
$$= \alpha E_t[\min\{\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i}), \alpha^n(z_{t+n+1} - \alpha^{L-1}h)\}]$$

But it is true for any sample path that

$$\min\{\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i}), \alpha^n(z_{t+n+1} - \alpha^{L-1}h)\} \le \min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i})$$

And so

$$E_t[\min\{\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i}), \alpha^n(z_{t+n+1} - \alpha^{L-1}h)\}] \le E_t[\min_{1\le i\le n} \alpha^{i-1}(z_{t+i} + H_{n-i})]$$

This establishes the inequality.

PROOF OF PROPOSITION 5. By induction. First show true for n=1 and $\forall t$.

$$R_{t,1} = \alpha E_t[z_{t+1}]$$
$$U_{t,1} = \alpha \min\{E_t[z_{t+1}] + H_0\} = \alpha E_t[z_{t+1}].$$

Therefore, $R_{t,1} \le U_{t,1}$ $\forall t$. Assume true for n and $\forall t$. It remains to show that $R_{t,n+1} \le U_{t,n+1}$ $\forall t$.

$R_{t,n+1} = \alpha E_t[\min(z_{t+1} + H_n, R_{t+1,n})]$ since the expectation of a minimum is less than or equal to the minimum of the expectations,

$$\le \alpha \min(E_t[z_{t+1}] + H_n, E_t[R_{t+1,n}])]$$

Now, $R_{t+1,n} \le U_{t+1,n} = \alpha \min_{1\le i\le n}\{\alpha^{i-1}(E_{t+1}[z_{t+1+i}] + H_{n-i})\}$ (by induction hypothesis)

$E_t[R_{t+1,n}] \le E_t\left[\alpha \min_{1\le i\le n}\{\alpha^{i-1}(E_{t+1}[z_{t+1+i}] + H_{n-i})\}\right]$ since the expectation of a minimum is less than or equal to the minimum of the expectations,

$$E_t[R_{t+1,n}] \le \alpha \min_{1\le i\le n}\left\{E_t\left[\alpha^{i-1}(E_{t+1}[z_{t+1+i}] + H_{n-i})\right]\right\}$$
$$= \alpha \min_{1\le i\le n}\{\alpha^{i-1}(E_t[z_{t+1+i}] + H_{n-i})\}$$

So, $R_{t,n+1} \le \alpha \min\left\{E_t[z_{t+1}] + H_n, \alpha \min_{1\le i\le n}\{\alpha^{i-1}(E_t[z_{t+1+i}] + H_{n-i})\}\right\}$ and

$$U_{t,n+1} = \alpha \min_{1\le i\le n+1}\{\alpha^{i-1}(E_t[z_{t+i}] + H_{n+1-i})\}$$

$$= \alpha \min\left\{ E_t[z_{t+1}] + H_n, \min_{2\le i\le n+1} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n+1-i}) \right\} \right\}$$

$$= \alpha \min\left\{ E_t[z_{t+1}] + H_n, \min_{1\le i\le n} \left\{ \alpha^{i}(E_t[z_{t+1+i}] + H_{n-i}) \right\} \right\}$$

Therefore, $R_{t,n+1} \le U_{t,n+1}$

PROOF OF PROPOSITION 6. $U_{t,n} - z_t - H_n \ge U_{t,n+1} - z_t - H_{n+1}$ iff

$U_{t,n+1} + H_n - H_{n+1} \le U_{t,n}$.

Now

$$U_{t,n} = \alpha \min_{1\le i\le n} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n-i}) \right\}$$

$$U_{t,n+1} + H_n - H_{n+1} = \alpha \min_{1\le i\le n+1} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n+1-i}) \right\} - \alpha^{L+n} h$$

$$= \alpha \min_{1\le i\le n+1} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n+1-i} - \alpha^{L+n-i} h) \right\}$$

$$= \min\left\{ \alpha \min_{1\le i\le n} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n+1-i} - \alpha^{L+n-i} h) \right\}, \alpha^{n+1}(E_t[z_{t+n+1}] - \alpha^{L-1} h) \right\}$$

$$= \min\left\{ \alpha \min_{1\le i\le n} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n-i}) \right\}, \alpha^{n+1}(E_t[z_{t+n+1}] - \alpha^{L-1} h) \right\}$$

$$= \min\left\{ U_{t,n}, \alpha^{n+1}(E_t[z_{t+n+1}] - \alpha^{L-1} h) \right\}$$

$$\le U_{t,n}$$

PROOF OF PROPOSITION 7.

$$\lim_{n\to\infty} U_{t,n} = \lim_{n\to\infty} \alpha \min_{1\le i\le n} \left\{ \alpha^{i-1}(E_t[z_{t+i}] + H_{n-i}) \right\}$$

$$= \lim_{n\to\infty} \min\left\{ \min_{1\le i\le n-1} \left\{ \alpha^{i}(E_t[z_{t+i}] + H_{n-i}) \right\}, \alpha^n E_t[z_{t+n}] \right\}$$

$$\le \lim_{n\to\infty} \alpha^n E_t[z_{t+n}] = 0$$

Since $z_t \ge 0 \;\; \forall t$, $U_{t,n} \ge 0 \;\; \forall t,n$. Therefore,

$\lim_{n\to\infty} U_{t,n} = 0$, and

$$\lim_{n\to\infty} S^{Max}_{t,n} = \lim_{n\to\infty} \left\{ U_{t,n} - z_t - H_n \right\} = -(z_t + H_\infty) \text{ where } H_\infty = \frac{\alpha^L}{1-\alpha} h$$

PROOF OF PROPOSITION 8. Immediately follows from proposition 7 since prices are nonnegative and $0 \le L_{t,n} \le R_{t,n} \le U_{t,n} \;\; \forall n, \forall t$