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PROBABILITY DENSITY FUNCTIONS OF ORDER STATISTICS OF NONIDENTICALLY DISTRIBUTED TRUNCATED CONTINUOUS RANDOM VARIABLES

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Abstract

In this paper, the theorems related to the probability density functions of order statistics of independent but not necessarily identically distributed truncated continuous random variables are given. Then some results are obtained.

Keywords: Order statistics, joint distribution, truncated distribution, permanent.

1. Introduction

A good deal of work has appeared in the literature about several identities and recurrence relations for probability density function (pdf) and distribution function (df) of order statistics of independent but not necessarily identically distributed (innid) random variables. [1] derived the joint pdf and marginal pdf of order statistics of innid random variables by means of permanents. [2] obtained the distribution of single order statistics in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2,..., X_n\}$ where X_i's are innid random variables. Several recurrence relations and identities for product moments of order statistics of innid random variables by using permanents were obtained [3]. Some expressions for the moments and product moments of the order statistics from the doubly truncated linear-exponential distributions were derived [4]. [5] obtained the recurrence relationships among the distribution function of order statistics arising from innid random variables. [6] generalized the independent and identically distributed (iid) results for the pareto and doubly-truncated pareto models established by [7]. [8] derived some general recurrence relations satisfied by single and product moments of order statistics from doubly truncated continuous distributions. [9] obtained the joint pdf and df of order statistics of innid random variables by means of permanents.

Let $F_1, F_2,...,F_n$ and $f_1, f_2,...,f_n$ be df and pdf of $X_1, X_2,…,X_n$, respectively. In addition, the inverse function of $F_1, F_2,...,F_n$ are expressed as

$$F_i^{-1}(p) = \inf\{x : F_i(x) \geq p\} = \sup\{x : F_i(x) < p\}, \tag{1}$$

where $x \in R,\ 0 < p < 1$ and $i = 1,2,…,n$ [10,11]. In (1), if $p = 0$ and then $p = 1$, the following equations are obtained [12].

$$\alpha(F_i) = \inf\{x : F_i(x) > 0\} = F_i^{-1}(0) \text{ and } w(F_i) = \sup\{x : F_i(x) < 1\} = F_i^{-1}(1). \tag{2}$$

From (2), we can write

$$\alpha({}_{uv}F_i) = u_i \text{ ve } w({}_{uv}F_i) = v_i. \tag{3}$$

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From (3), df and pdf of X_i which truncated on the left at u_i and right at v_i, respectively, are expressed as

$$ {}_{uv}F_i(x) = \frac{F_i(x) - F_i(u_i)}{F_i(v_i) - F_i(u_i)} \quad \text{and} \quad {}_{uv}f_i(x) = \frac{f_i(x)}{F_i(v_i) - F_i(u_i)} . \tag{4} $$

Let $X_1, X_2, \ldots, X_n$ be innid continuous random variables which are truncated and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the corresponding order statistics.

In addition, $X^s{}_{1:n}, X^s{}_{2:n}, \ldots, X^s{}_{n:n}$ are order statistics of truncated random variables with common df ${}_{uv}F^s$ and pdf ${}_{uv}f^s$, respectively, defined by

$$ {}_{uv}F^s = \frac{1}{n_s} \sum_{i \in s} {}_{uv}F_i \tag{5} $$

and

$$ {}_{uv}f^s = \frac{1}{n_s} \sum_{i \in s} {}_{uv}f_i \tag{6} $$

where *s* is a subset of integers $\{1,2,\ldots,n\}$ with $n_s \ge 1$ elements. Moreover, if $a_1, a_1, \cdots$ are column vectors, then $\underset{i_1 \ \ i_2 \ \ \cdots}{A = [a_1, a_2, \ldots]}$ will denote the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on. Finally, Per(A) denotes the permanent of a square matrix A, which is defined similar to the determinants except that all terms in the expansion have a positive sign [13].

2. Main Theorems

In this section, we obtain theorems related to the probability density functions of order statistics of innid truncated continuous random variables. Now the pdf of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_d:n}$ $(1 \le r_1 < r_2 < \ldots < r_d \le n,$ $d = 1,2,\ldots,n)$ will be given.

Theorem 2.1.

$$ f_{r_1, r_2, \ldots, r_d : n}(x_1, x_2, \ldots, x_d) = D \sum_{t_d, \ldots, t_2, t_1}^{n, \ldots, r_3 - 1, r_2 - 1} (-1)^{-d + \sum_{w=1}^{d} (r_{w+1} - t_w)} \prod_{w=1}^{d} \binom{r_{w+1} - r_w - 1}{t_w - r_w} $$
$$ \sum_{n_\varsigma = n + r_d - t_d} (t_d - r_d)! \sum_{n_{\varsigma_1}, n_{\varsigma_2}, \ldots, n_{\varsigma_{d-1}}} \prod_{w=1}^{d} \sum_{n_{\varsigma_w}} per[\underset{r_{w+1} - r_{w-1} - 1 - t_w + t_{w-1}}{{}_{uv}\mathrm{F}(x_w)}][\varsigma_w / .)\, per[\underset{1}{{}_{uv}\mathrm{f}(x_w)}][\varsigma'_w / .) \tag{7} $$

where $D = [\prod_{w=1}^{d+1} (r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$, $r_{d+1} = n + 1$, $\sum_{t_d, \ldots, t_2, t_1}^{n, \ldots, r_3 - 1, r_2 - 1} = \sum_{t_d = r_d}^{n} \cdots \sum_{t_2 = r_2}^{r_3 - 1} \sum_{t_1 = r_1}^{r_2 - 1}$, $\varsigma = \bigcup_{w=1}^{d} \varsigma_w$, $\varsigma \subset N$, $\upsilon \ne v$, $\varsigma_\upsilon \bigcap \varsigma_v = \phi$, $s = \varsigma_w \bigcup \varsigma'_w$, $n_{\varsigma_w} = r_{w+1} - r_{w-1} - t_w + t_{w-1}$, $t_0 = r_1 - 1$, $n_{\varsigma_w} = r_{w+1} - r_{w-1} - 1 - t_w + t_{w-1}$ and $n_{\varsigma'_w} = 1$. In addition, ${}_{uv}\mathrm{f}(x_w) = ({}_{uv}f_1(x_w), {}_{uv}f_2(x_w), \ldots, {}_{uv}f_n(x_w))'$ is column vector.

Proof. Let us consider

$$P\{x_1 < X_{r_1:n} \le x_1 + \delta x_1, x_2 < X_{r_2:n} \le x_2 + \delta x_2, \ldots, x_d < X_{r_d:n} \le x_d + \delta x_d\} \tag{8}$$

Dividing (8) to $\prod_{w=1}^{d} \delta x_w$ and $\delta x_1 \to 0, \delta x_2 \to 0, \ldots, \delta x_d \to 0$ we obtain

$${}_{uv}f_{r_1,r_2,\ldots,r_d:n}(x_1, x_2, \ldots, x_d) = D\, per\mathrm{B} \tag{9}$$

where $\mathrm{B} = [\underset{r_1-1}{{}_{uv}\mathrm{F}(x_1)}\ \underset{1}{{}_{uv}\mathrm{f}(x_1)}\ \underset{r_2-r_1-1}{{}_{uv}\mathrm{F}(x_2) - {}_{uv}\mathrm{F}(x_1)}\ \underset{1}{{}_{uv}\mathrm{f}(x_2)} \ldots \underset{1}{{}_{uv}\mathrm{f}(x_d)}\ \underset{n-r_d}{1 - {}_{uv}\mathrm{F}(x_d)}]$ is a square matrix.

Using properties of permanent, it can be written

$$per\mathrm{B} = per[\underset{r_1-1}{{}_{uv}\mathrm{F}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_1)}\quad \underset{r_2-r_1-1}{{}_{uv}\mathrm{F}(x_2) - {}_{uv}\mathrm{F}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_2)} \ \cdots \ \underset{1}{{}_{uv}\mathrm{f}(x_d)}\quad \underset{n-r_d}{1 - {}_{uv}\mathrm{F}(x_d)}]$$

$$= \sum_{t_d=0}^{n-r_d} (-1)^{n-r_d-t_d} \binom{n-r_d}{t_d} \cdots \sum_{t_2=0}^{r_3-r_2-1} (-1)^{r_3-r_2-1-t_2} \binom{r_3-r_2-1}{t_2} \sum_{t_1=0}^{r_2-r_1-1} (-1)^{r_2-r_1-1-t_1} \binom{r_2-r_1-1}{t_1}$$

$$.per[\underset{r_2-2-t_1}{{}_{uv}\mathrm{F}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_1)}\quad \underset{r_3-r_2-1-t_2+t_1}{{}_{uv}\mathrm{F}(x_2)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_2)} \ \cdots \ \underset{1}{{}_{uv}\mathrm{f}(x_d)}\quad \underset{t_d}{1}\quad \underset{n-r_d-t_d+t_{d-1}}{{}_{uv}\mathrm{F}(x_d)}\ ]$$

$$= \sum_{t_d=0}^{n-r_d} \cdots \sum_{t_2=0}^{r_3-r_2-1} \sum_{t_1=0}^{r_2-r_1-1} (-1)^{n+1-r_1-d-\sum_{w=1}^{d} t_w} \prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w} \sum_{n_\varsigma = n-t_d} t_d!$$

$$.per[\underset{r_2-2-t_1}{{}_{uv}\mathrm{F}(x_1)}\quad \underset{r_3-r_2-1-t_2+t_1}{{}_{uv}\mathrm{F}(x_2)} \ \cdots \ \underset{n-r_d-t_d+t_{d-1}}{{}_{uv}\mathrm{F}(x_d)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_2)} \cdots \underset{1}{{}_{uv}\mathrm{f}(x_d)}][\varsigma/.)$$

$$= \sum_{t_d=r_d}^{n} \cdots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \sum_{n_\varsigma = n+r_d-t_d} (t_d-r_d)!$$

$$.per[\underset{r_2+r_1-2-t_1}{{}_{uv}\mathrm{F}(x_1)}\quad \underset{r_3-r_1-1-t_2+t_1}{{}_{uv}\mathrm{F}(x_2)} \ \cdots \ \underset{n-r_{d-1}-t_d+t_{d-1}}{{}_{uv}\mathrm{F}(x_d)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_2)} \cdots \underset{1}{{}_{uv}\mathrm{f}(x_d)}][\varsigma/.)$$

$$= \sum_{t_d=r_d}^{n} \cdots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \sum_{n_\varsigma = n+r_d-t_d} (t_d-r_d)! \sum_{n_{\varsigma_1}, n_{\varsigma_2}, \ldots, n_{\varsigma_{d-1}}}$$

$$.per[\underset{r_2+r_1-2-t_1}{{}_{uv}\mathrm{F}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_1)}][\varsigma_1/.)\ per[\underset{r_3-r_1-1-t_2+t_1}{{}_{uv}\mathrm{F}(x_2)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_2)}][\varsigma_2/.) \ldots per[\underset{n-r_{d-1}-t_d+t_{d-1}}{{}_{uv}\mathrm{F}(x_d)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_d)}][\varsigma_d/.)$$

$$= \sum_{t_d=r_d}^{n} \cdots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \sum_{n_\varsigma = n+r_d-t_d} (t_d-r_d)! \sum_{n_{\varsigma_1}, n_{\varsigma_2}, \ldots, n_{\varsigma_{d-1}}}$$

$$.\prod_{w=1}^{d} per[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{{}_{uv}\mathrm{F}(x_w)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_w)}][\varsigma_w/.)$$

$$= \sum_{t_d=r_d}^{n} \cdots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w-r_w} \sum_{n_\varsigma = n+r_d-t_d} (t_d-r_d)! \sum_{n_{\varsigma_1}, n_{\varsigma_2}, \ldots, n_{\varsigma_{d-1}}}$$

$$\prod_{w=1}^{d}\sum_{n_{\varsigma_w}} per[\ {}_{uv}\mathrm{F}(x_w)_{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}\ ][\varsigma_w/.)\ per[{}_{uv}\mathrm{f}(x_w)_1][\varsigma'_w/.) \quad . \tag{10}$$

After substituting (10) into (9), we can obtained (7).

Theorem 2.1. can be expressed as Theorem 2.2. using properties of permanent.

Theorem 2.2.

$${}_{uv}f_{r_1,r_2,...,r_d:n}(x_1,x_2,\dots,x_d) = D\sum_{t_d,...,t_2,t_1}^{n,...,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}$$

$$\sum_{n_\varsigma=n+r_d-t_d}(t_d-r_d)!\sum_{n_{\varsigma_1},n_{\varsigma_2},...,n_{\varsigma_{d-1}}}\prod_{w=1}^{d}\sum_{n_{\varsigma_w}} n_{\varsigma_w}!\prod_{l=1}^{n_{\varsigma_w}}{}_{uv}F_{\varsigma_w^l}(x_w)\ {}_{uv}f_{\varsigma_w'^w}(x_w) \quad . \tag{11}$$

where $\varsigma_w=\{\varsigma_w^1,\varsigma_w^2,\dots,\varsigma_w^{n_{\varsigma_w}}\}$ and $\varsigma'_w=\{\varsigma_w'^w\}$.

Theorem 2.2. can be expressed as Theorem 2.3. using (5) and (6).

Theorem 2.3.

$${}_{uv}f_{r_1,r_2,...,r_d:n}(x_1,x_2,\dots,x_d) = \sum\sum n!\,D\sum_{t_d,...,t_2,t_1}^{n,...,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}$$

$$.\ [{}_{uv}F^s(x_w)]^{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}\ {}_{uv}f^s(x_w) \tag{12}$$

where $\sum\sum=\sum_{\kappa=1}^{n}(-1)^{n-\kappa}\frac{\kappa^n}{n!}\sum_{n_s=\kappa}$.

Proof. (8) can be expressed as

$$\sum\sum P\{x_1<X^s_{r_1:n}\le x_1+\delta x_1,x_2<X^s_{r_2:n}\le x_2+\delta x_2,\dots,x_d<X^s_{r_d:n}\le x_d+\delta x_d\} \quad . \tag{13}$$

Dividing (13) to $\prod_{w=1}^{d}\delta x_w$ and $\delta x_1\to 0,\delta x_2\to 0,\dots,\delta x_d\to 0$ we obtain (12).

Theorem 2.1. can be expressed as Theorem 2.4. using expansion of permanent.

Theorem 2.4.

$${}_{uv}f_{r_1,r_2,...,r_d:n}(x_1,x_2,\dots,x_d) = D\sum_P\prod_{l=1}^{r_1-1}{}_{uv}F_{j_l}(x_1)$$

$$\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\sum_{\substack{n_\tau=t-r_{w-1}\\ n_{\tau'}=r_w-1-t}}\prod_{l=1}^{t-r_{w-1}}{}_{uv}F_{\tau_l}(x_w)\prod_{l=1}^{r_w-1-t}{}_{uv}F_{\tau'_l}(x_{w-1})\prod_{w=1}^{d}{}_{uv}f_{j_{r_w}}(x_w) \quad . \tag{14}$$

where $\sum_P$ denotes the sum over all $n!$ permutations $\{j_1,j_2,\dots,j_n\}$ of $\{1,2,\dots,n\}$, $\tau=\{\tau_1,\tau_2,\dots,\tau_{t-r_{w-1}}\}$, $\tau'=\{\tau'_1,\tau'_2,\dots,\tau'_{r_w-t-1}\}$, $\tau\cap\tau'=\phi$ and $\tau\cup\tau'=\{j_{r_{w-1}+1},j_{r_{w-1}+2},\dots,j_{r_w-1}\}$.

Proof. Let us consider

$$P\{x_1 < X_{r_1:n} \le x_1 + \delta x_1, x_2 < X_{r_2:n} \le x_2 + \delta x_2, \ldots, x_d < X_{r_d:n} \le x_d + \delta x_d\} \quad (15)$$

Dividing (15) to $\prod_{w=1}^{d} \delta x_w$ and $\delta x_1 \to 0, \delta x_2 \to 0, \ldots, \delta x_d \to 0$ we obtain

$$_{uv}f_{r_1,r_2,\ldots,r_d:n}(x_1, x_2, \ldots, x_d) = D \sum_{p} \prod_{l=1}^{r_1-1} [_{uv}F_{j_l}(x_1)]\, _{uv}f_{j_{r_1}}(x_1) \prod_{l=r_1+1}^{r_2-1} [_{uv}F_{j_l}(x_2) - {}_{uv}F_{j_l}(x_1)]$$

$$_{uv}f_{j_{r_2}}(x_2) \ldots {}_{uv}f_{j_{r_d}}(x_d) \prod_{l=r_d+1}^{n} [1 - {}_{uv}F_{j_l}(x_d)] \quad (16)$$

(16) can be written,

$$_{uv}f_{r_1,r_2,\ldots,r_d:n}(x_1, x_2, \ldots, x_d) = D \sum_{P} \prod_{l=1}^{r_1-1} {}_{uv}F_{j_l}(x_1) \prod_{w=2}^{d+1} \prod_{l=r_{w-1}+1}^{r_w-1} [_{uv}F_{j_l}(x_w) - {}_{uv}F_{j_l}(x_{w-1})] \prod_{w=1}^{d} {}_{uv}f_{j_{r_w}}(x_w) \quad (17)$$

Now, let us consider the following equation

$$\prod_{l=r_{w-1}+1}^{r_w-1} [_{uv}F_{j_l}(x_w) - {}_{uv}F_{j_l}(x_{w-1})] = \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \sum_{\substack{n_\tau = t-r_{w-1} \\ n_{\tau'} = r_w-1-t}} \prod_{l=1}^{t-r_{w-1}} {}_{uv}F_{\tau_l}(x_w) \prod_{l=1}^{r_w-1-t} {}_{uv}F_{\tau'_l}(x_{w-1}) \quad (18)$$

After substituting (18) into (17), we can obtained (14). Theorem 2.4. can be expressed as Theorem 2.5.

Theorem 2.5.

$$_{uv}f_{r_1,r_2,\ldots,r_d:n}(x_1, x_2, \ldots, x_d) = D^{-1} \sum_{_D P_{r_d,\ldots,r_2,r_1}} \prod_{l=1}^{r_1-1} {}_{uv}F_{j_l}(x_1)$$

$$\prod_{w=2}^{d+1} \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \sum_{\substack{n_\tau = t-r_{w-1} \\ n_{\tau'} = r_w-1-t}} \prod_{l=1}^{t-r_{w-1}} {}_{uv}F_{\tau_l}(x_w) \prod_{l=1}^{r_w-1-t} {}_{uv}F_{\tau'_l}(x_{w-1}) \prod_{w=1}^{d} {}_{uv}f_{j_{r_w}}(x_w) \quad (19)$$

where $\sum_{_D P_{r_d,\ldots,r_2,r_1}}$ denotes the sum over all permutations $(j_1, j_2, \ldots, j_n)$ of $(1,2,\ldots,n)$ for $j_1 < j_2 < \ldots < j_{r_1-1}$, $j_{r_1+1} < j_{r_1+2} < \ldots < j_{r_2-1}$, …, $j_{r_d+1} < j_{r_d+2} < \ldots < j_n$.

Theorem 2.4. can be expressed as Theorem 2.6.

Theorem 2.6.

$$_{uv}f_{r_1,r_2,\ldots,r_d:n}(x_1, x_2, \ldots, x_d) = (n - r_d)! \sum_{P_{r_d}} \prod_{l=1}^{r_1-1} {}_{uv}F_{j_l}(x_1)$$

$$\prod_{w=2}^{d+1} \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \sum_{\substack{n_\tau = t-r_{w-1} \\ n_{\tau'} = r_w-1-t}} \prod_{l=1}^{t-r_{w-1}} {}_{uv}F_{\tau_l}(x_w) \prod_{l=1}^{r_w-1-t} {}_{uv}F_{\tau'_l}(x_{w-1}) \prod_{w=1}^{d} {}_{uv}f_{j_{r_w}}(x_w) \quad (20)$$

where $\sum_{P_{r_d}}$ denotes the sum over all permutations $\{j_1, j_2, \ldots, j_{r_d}\}$ of $\{1,2,\ldots,n\}$.

Theorem 2.4. can be expressed as Theorem 2.7. using (5) and (6).

Theorem 2.7.

$$
{}_{uv}f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=\sum\sum n!D[{}_{uv}F^s(x_1)]^{r_1-1} \cdot\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\binom{r_w-r_{w-1}-1}{t-r_{w-1}}[{}_{uv}F^s(x_w)]^{t-r_{w-1}}[{}_{uv}F^s(x_{w-1})]^{r_w-1-t}\prod_{w=1}^{d}{}_{uv}f^s(x_w) \quad (21)
$$

3. Results

In this section, we derive some results for the probability density functions of order statistics of innid truncated continuous random variables. We will express the following result for pdf of the *r*th order statistics from innid variables from truncated distribution.

Result 3.1.

$$
{}_{uv}f_{r:n}(x)=\frac{1}{(r-1)!(n-r)!}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}\sum_{n_{\varsigma}=n+r-t}(t-r)! \cdot\sum_{n_{\varsigma}=n+r-1-t} per[{}_{uv}\mathrm{F}(x)][\varsigma/.]_{n+r-1-t}\, per[{}_{uv}\mathrm{f}(x)][\varsigma'/.]_{1}
$$

$$
=\frac{1}{(r-1)!(n-r)!}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}\sum_{n_{\varsigma}=n+r-t}(t-r)!\sum_{n_{\varsigma}=n+r-1-t}(n+r-1-t)!\prod_{l=1}^{n+r-1-t}{}_{uv}F_{\varsigma^l}(x)\,{}_{uv}f_{\varsigma'}(x)
$$

$$
=\sum\sum r\binom{n}{r}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}[{}_{uv}F^s(x)]^{n+r-1-t}{}_{uv}f^s(x)
$$

$$
=\frac{1}{(r-1)!(n-r)!}\sum_{P}\prod_{l=1}^{r-1}{}_{uv}F_{j_l}(x)\sum_{t=r}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}{}_{uv}F_{\tau'_l}(x)\,{}_{uv}f_{j_r}(x)
$$

$$
=(r-1)!(n-r)!\sum_{D\,P_r}\prod_{l=1}^{r-1}{}_{uv}F_{j_l}(x)\sum_{t=r}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}{}_{uv}F_{\tau'_l}(x)\,{}_{uv}f_{j_r}(x)
$$

$$
=(n-r)!\sum_{P_r}\prod_{l=1}^{r-1}{}_{uv}F_{j_l}(x)\sum_{t=r}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}{}_{uv}F_{\tau'_l}(x)\,{}_{uv}f_{j_r}(x)
$$

$$
=\sum\sum r\binom{n}{r}[{}_{uv}F^s(x)]^{r-1}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}[{}_{uv}F^s(x)]^{n-t}{}_{uv}f^s(x) \quad (22)
$$

Proof. In (7), (11), (12), (14), (19), (20) and (21), if $d=1$, (22) is obtained.

We will express the following result for pdf of the minimum order statistics from innid variables from truncated distribution.

Result 3.2.

$$
{}_{uv}f_{1:n}(x)=\frac{1}{(n-1)!}\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}\sum_{n_{\varsigma}=n+1-t}(t-1)!\sum_{n_{\varsigma}=n-t} per[{}_{uv}\mathrm{F}(x)][\varsigma/.]_{n-t}\, per[{}_{uv}\mathrm{f}(x)][\varsigma'/.]_{1}
$$

$$= \frac{1}{(n-1)!}\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}\sum_{n_{\varsigma}=n+1-t}(t-1)!\sum_{n_{\varsigma}=n-t}(n-t)!\prod_{l=1}^{n-t}{}_{uv}F_{\varsigma^{l}}(x)\,{}_{uv}f_{\varsigma'}(x)$$

$$= \sum\sum n\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}[{}_{uv}F^{s}(x)]^{n-t}{}_{uv}f^{s}(x)$$

$$= \frac{1}{(n-1)!}\sum_{P}\sum_{t=1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}{}_{uv}F_{\tau'_{l}}(x)_{uv}f_{j_1}(x)$$

$$= (n-1)!\sum_{{}_{D}P_1}\sum_{t=1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}{}_{uv}F_{\tau'_{l}}(x)_{uv}f_{j_1}(x)$$

$$= (n-1)!\sum_{P_1}\sum_{t=1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}{}_{uv}F_{\tau'_{l}}(x)_{uv}f_{j_1}(x) \quad . \tag{23}$$

Proof. In (22), if $r=1$, (23) is obtained.

We will express the following result for pdf of the maximum order statistics from innid variables from truncated distribution.

Result 3.3.

$${}_{uv}f_{n:n}(x) = \frac{1}{(n-1)!}\sum_{n_{\varsigma}=n-1} per[\underset{n-1}{{}_{uv}\mathrm{F}(x)}][\varsigma/.)\,per[\underset{1}{{}_{uv}\mathrm{f}(x)}][\varsigma'/.)$$

$$= \sum_{n_{\varsigma}=n-1}\prod_{l=1}^{n-1}{}_{uv}F_{\varsigma^{l}}(x)\,{}_{uv}f_{\varsigma'}(x)$$

$$= \sum\sum n\,[{}_{uv}F^{s}(x)]^{n-1}{}_{uv}f^{s}(x)$$

$$= \frac{1}{(n-1)!}\sum_{P}\prod_{l=1}^{n-1}{}_{uv}F_{j_l}(x)_{uv}f_{j_n}(x)$$

$$= (n-1)!\sum_{{}_{D}P_n}\prod_{l=1}^{n-1}{}_{uv}F_{j_l}(x)_{uv}f_{j_n}(x)$$

$$= \sum_{P_n}\prod_{l=1}^{n-1}{}_{uv}F_{j_l}(x)_{uv}f_{j_n}(x) \quad . \tag{24}$$

Proof. In (22), if $r=1$, (24) is obtained.

We will express the following result for the joint pdf of $X_{1:n}$ and $X_{n:n}$ of the order statistics from innid variables from truncated distribution.

Result 3.4.

$${}_{uv}f_{1,n:n}(x_1,x_2) = \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\binom{n-2}{t-1}\sum_{n_{\varsigma_1}=n-t}\prod_{w=1}^{2}\sum_{n_{\varsigma_w}} per[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{{}_{uv}\mathrm{F}(x_w)}][\varsigma_w/.)\,per[\underset{1}{{}_{uv}\mathrm{f}(x_w)}][\varsigma'_w/.)$$

$$= \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\binom{n-2}{t-1}\sum_{n_{\varsigma_1}=n-t}\prod_{w=1}^{2}\sum_{n_{\varsigma_w}} n_{\varsigma_w}!\prod_{l=1}^{n_{\varsigma_w}}{}_{uv}F_{\varsigma_w^{l}}(x_w)\,{}_{uv}f_{\varsigma'^{w}_{w}}(x_w)$$

$$=\sum\sum n(n-1)\sum_{t=1}^{n-1}(-1)^{n-1-t}\binom{n-2}{t-1}[{}_{uv}F^{s}(x_1)]^{n-1-t}[{}_{uv}F^{s}(x_2)]^{t-1}{}_{uv}f^{s}(x_1)\,{}_{uv}f^{s}(x_2)$$

$$=\frac{1}{(n-2)!}\sum_{P}\sum_{t=1}^{n-1}(-1)^{n-1-t}\sum_{\substack{n_\tau=t-1\\ n_{\tau'}=n-1-t}}\prod_{l=1}^{t-1}{}_{uv}F_{\tau_l}(x_2)\prod_{l=1}^{n-1-t}{}_{uv}F_{\tau'_l}(x_1)_{uv}f_{j_1}(x_1)_{uv}f_{j_n}(x_2)$$

$$=(n-2)!\sum_{{}_D P_{n,1}}\sum_{t=1}^{n-1}(-1)^{n-1-t}\sum_{\substack{n_\tau=t-1\\ n_{\tau'}=n-1-t}}\prod_{l=1}^{t-1}{}_{uv}F_{\tau_l}(x_2)\prod_{l=1}^{n-1-t}{}_{uv}F_{\tau'_l}(x_1)_{uv}f_{j_1}(x_1)_{uv}f_{j_n}(x_2)$$

$$=\sum_{P_n}\sum_{t=1}^{n-1}(-1)^{n-1-t}\sum_{\substack{n_\tau=t-1\\ n_{\tau'}=n-1-t}}\prod_{l=1}^{t-1}{}_{uv}F_{\tau_l}(x_2)\prod_{l=1}^{n-1-t}{}_{uv}F_{\tau'_l}(x_1)_{uv}f_{j_1}(x_1)_{uv}f_{j_n}(x_2)\,. \qquad (25)$$

Proof. In (7), (11), (12), (14), (19), (20) and (21), if $d=2$, $r_1=1$ and $r_2=n$, (25) is obtained.

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