JESTECH 16(1), 17-21, (2013)

Engineering Science and Technology, an International Journal (JESTECH)

journal homepage: jestech.karabuk.edu.tr

On joint distributions of order statistics from innid weibull variables

Gökhan Gökdere*

*Bingol University, Department of Mathematics, Bingol, Turkey

ARTICLE INFO

Article history:
Received 1 March 2013
Accepted 29 March 2013

Keywords:
Order statistics,
Innid random variable,
Joint distribution,
Weibull distribution

ABSTRACT

In this paper, the joint distributions of order statistics from innid variables from a Weibull distribution are obtained. Some results related to the distribution and probability density functions of order statistics from innid Weibull variables are also given.

1. Introduction

There is a broad literature on order statistics. Recently, numerous authors have established several identities and recurrence relations for probability density function(pdf) and distribution function(df) of order statistics of independent and identically distributed(iid) random variables. For more details, see Arnold et al [1], Balasubramanian and Beg [2] , Reiss [3] and David [4]. Furthermore, some well-known recurrence relations for order statistics in the *iid* case are generalized to the case when the variables are independent but not necessarily identically distributed(innid) in Balakrishnan [5]. Vaughan and Venables[6] derived the joint pdf and marginal pdf of order statistics of innid random variables by means of permanents. Balasubramanian et al [7] obtained the distribution of single order statistics in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2,..., X_n\}$ where X_i 's are innid random variables. Balakrishnan [8] and Güngör [9] obtained the joint pdf and df of order statistics of innid random variables by means of permanents. The joint distributions of order statistics from innid variables from a truncated distribution are obtained by Güngör [10]. The theorems related to the pdf of order statistics of innid truncated continuous random variables are given by Gökdere [11]. Malik and Trudel [12] used the Mellin transform technique, to find the distribution of the *i*th and *j*th order statistics from the Pareto, Power and Weibull distributions. Recurrence relations for the single and product moments of order statistics n non-identically distributed doubly truncated generalized Weibull random variables are established by Afify [13].

In this paper we consider the case where the random variables $X_{r_k:n}, X_{r_{k+1}:n},..., X_{r_{k+d-1}:n}$ ($1 \le r_k < r_{k+1} < ... < r_{k+d-1} \le n$, $d = 1,2,..., n$ and $k = 1,2,..., n-d+1$).are independent and not necessarily identical having Weibull distributions with pdf

$$f_{j_i}(x) = \alpha_{j_i} \beta x^{\beta-1} e^{-\alpha_{j_i} x^{\beta}}, \ x \in (0,\infty), \alpha_{j_i}, \beta > 0, \qquad (1)$$

and df

$$F_{j_i}(x) = 1 - e^{-\alpha_{j_i} x^{\beta}}, \ x \in (0,\infty), \ \alpha_{j_i}, \beta > 0, \qquad (2)$$

for $i = 1,2,..., n$. The paper is organized as follows. In section 2, we give the pdf of $X_{r_k:n}, X_{r_{k+1}:n},..., X_{r_{k+d-1}:n}$. Then the pdf of the *r*th order statistics from innid variables from Weibull distribution is obtained. In section 3, at first we give the df of $X_{r_k:n}, X_{r_{k+1}:n},..., X_{r_{k+d-1}:n}$ and then we obtain the *r*th order statistics from innid variables from Weibull distribution.

* Corresponding author. *E-mail address: g.g.gokdere@gmail.com* (G. Gökdere).

2. The pdf of Order Statistics from innid Variables

In this section, we use Weibull distribution in (1) and (2) to establish the joint pdf of $X_{r_k:n}, X_{r_{k+1}:n},\ldots, X_{r_{k+d-1}:n}$. Then the pdf of the *r*th order statistics from innid variables from Weibull distribution is obtained.

Theorem 2.1.

$$f_{r_k,r_{k+1},\ldots,r_{k+d-1}:n}(x_k, x_{k+1},\ldots, x_{k+d-1}) = D\beta^d \sum_P \left[\prod_{w=k}^{k+d} \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \sum_{\substack{n_\tau=t-r_{w-1}\\ n_{\tau'}=r_w-1-t}} e^{-(\mathrm{A}+\mathrm{B}+\mathrm{C})} \right] \prod_{w=k}^{k+d-1} \alpha_{j_{r_w}} x_w^{\beta-1}$$

where $D = \left(\prod_{w=k}^{k+d} (r_w - r_{w-1} - 1)! \right)^{-1}$, $r_{k-1} = 0$, $r_{k+d} = n+1$, $\sum_P$ denotes the summation over all $n!$ permutations $\{j_1, j_2,\ldots, j_n\}$ of $\{1,2,\ldots,n\}$. $\sum_{\substack{n_\tau=t-r_{w-1}\\ n_{\tau'}=r_w-1-t}}$ denotes the sum over all $\binom{r_w - r_{w-1} - 1}{t - r_{w-1}}$ subsets $\tau = \{\tau_1, \tau_2,\ldots, \tau_{t-r_{w-1}}\}$ and $\tau' = \{\tau'_1, \tau'_2,\ldots, \tau'_{r_w-1-t}\}$ of $\{j_{r_{w-1}+1}, j_{r_{w-1}+2},\ldots, j_{r_w-1}\}$, $\mathrm{A} = \sum_{l=1}^{t-r_{w-1}} \alpha_{\tau_l} x_w^\beta$, $\mathrm{B} = \sum_{l=1}^{r_w-1-t} \alpha_{\tau'_l} x_{w-1}^\beta$ and $\mathrm{C} = \sum_{w=k}^{k+d-1} \alpha_{j_{r_w}} x_w^\beta$.

Proof. Let us consider,

$$\Pr\{x_k < X_{r_k:n} \le x_k + \delta x_k, x_{k+1} < X_{r_{k+1}:n} \le x_{k+1} + \delta x_{k+1},\ldots, x_{k+d-1} < X_{r_{k+d-1}:n} \le x_{k+d-1} + \delta x_{k+d-1}\} \tag{3}$$

Dividing (3) to $\prod_{w=k}^{k+d-1} \delta x_w$ and $\delta x_k \to 0$, $\delta x_{k+1} \to 0$,…, $\delta x_{k+d-1} \to 0$ we obtain

$$\begin{aligned} f_{r_k,r_{k+1},\ldots,r_{k+d-1}:n}(x_k, x_{k+1},\ldots, x_{k+d-1}) &= \left(\prod_{w=k}^{k+d} (r_w - r_{w-1} - 1)! \right)^{-1} \sum_P \left[\prod_{l=1}^{r_k-1} [F_{j_l}(x_k)] \right] f_{j_{r_k}}(x_k) \\ &\left[\prod_{l=r_k+1}^{r_{k+1}-1} [F_{j_l}(x_{k+1}) - F_{j_l}(x_k)] \right] f_{j_{r_{k+1}}}(x_{k+1}) \ldots f_{j_{r_{k+d-1}}}(x_{k+d-1}) \left[\prod_{l=r_{k+d-1}+1}^{n} [1 - F_{j_l}(x_{k+d-1})] \right] \end{aligned} \tag{4}$$

(4) can be written as,

$$\begin{aligned} f_{r_k,r_{k+1},\ldots,r_{k+d-1}:n}(x_k, x_{k+1},\ldots, x_{k+d-1}) &= \left(\prod_{w=k}^{k+d} (r_w - r_{w-1} - 1)! \right)^{-1} \\ &\sum_P \left[\prod_{w=k}^{k+d} \prod_{l=r_{w-1}+1}^{r_w-1} [F_{j_l}(x_w) - F_{j_l}(x_{w-1})] \right] \prod_{w=k}^{k+d-1} f_{j_{r_w}}(x_w) \end{aligned} \tag{5}$$

where $F_{j_i}(x_{k-1}) = 0$ and $F_{j_i}(x_{k+d}) = 1$.

Now, consider (2) it follows that

$$\prod_{l=r_{w-1}+1}^{r_w-1} \left[e^{-\alpha_{j_l} x_{w-1}^\beta} - e^{-\alpha_{j_l} x_w^\beta} \right] = \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \sum_{\substack{n_\tau=t-r_{w-1}\\ n_{\tau'}=r_w-1-t}} e^{-\left(\sum_{l=1}^{t-r_{w-1}} \alpha_{\tau_l} x_w^\beta + \sum_{l=1}^{r_w-1-t} \alpha_{\tau'_l} x_{w-1}^\beta \right)}.$$

If above relation and (1) are used in the (5), we get the proof.

In the following theorem we will give the pdf of $X_{r_k:n}$ ($1 \le r_k \le n$ and $k = 1,2,\ldots,n$).

Theorem 2.2.

$$f_{r_k:n}(x)=\frac{\beta x^{\beta-1}}{(r_k-1)!(n-r_k)!}\sum_P \alpha_{j_{r_k}}\sum_{t=1}^{r_k}(-1)^{r_k-t}\sum_{\substack{n_\tau=t-1\\ n_{\tau'}=r_k-t}} e^{-\left(\mathrm{D}+E+\alpha_{j_{r_k}}x^\beta\right)} \tag{6}$$

where, $\sum_{\substack{n_\tau=t-1\\ n_{\tau'}=r_k-t}}$ denotes the sum over all $\binom{r_k-1}{t-1}$ subsets $\tau=\{\tau_1,\tau_2,...,\tau_{t-1}\}$ and $\tau'=\{\tau'_1,\tau'_2,...,\tau'_{r_k-t}\}$ of $\{j_1,j_2,...,j_{r_k-1}\}$, $\mathrm{D}=\sum_{l=1}^{r_k-t}\alpha_{\tau_l}x^\beta$ and $\mathrm{E}=\sum_{l=r_k+1}^{n}\alpha_{j_l}x^\beta$.

Proof. In (5), if $d=1$, we can obtain as

$$f_{r_k:n}(x)=\frac{1}{(r_k-1)!(n-r_k)!}\sum_P\left[\prod_{l=1}^{r_k-1}F_{j_l}(x)\right]\left[\prod_{l=r_k+1}^{n}[1-F_{j_l}(x)]\right]f_{j_{r_k}}(x).$$

Now, consider (1) and (2) it follows that

$$f_{r_k:n}(x)=\frac{\beta x^{\beta-1}}{(r_k-1)!(n-r_k)!}\sum_P\alpha_{j_{r_k}}\prod_{l=1}^{r_k-1}[1-e^{-\alpha_{j_l}x^\beta}]e^{-\left(\sum_{l=r_k+1}^{n}\alpha_{j_l}+\alpha_{j_{r_k}}\right)x^\beta}.$$

Upon using

$$\prod_{l=1}^{r_k-1}[1-e^{-\alpha_{j_l}x^\beta}]=\sum_{t=1}^{r_k}(-1)^{r_k-t}\sum_{\substack{n_\tau=t-1\\ n_{\tau'}=r_k-t}}\prod_{l=1}^{r_k-t}e^{-\alpha_{\tau_l}x^\beta}$$

we get the proof.

Putting $r_k=1$ and $r_k=n$ in (6), respectively, we obtain the pdf's of minimum and maximum order statistics from non-identically distributed continuous weibull variables.

3. The df of Order Statistics from innid Variables

In this section, we again use Weibull distribution in (1) and (2) to establish the joint df of $X_{r_k:n},X_{r_{k+1}:n},...,X_{r_{k+d-1}:n}$. Then we obtain the *r*th order statistics from innid variables from Weibull distribution.

Theorem 3.1.

$$F_{r_k,r_{k+1},...r_{k+d-1}:n}(x_k,x_{k+1},...,x_{k+d-1})=\sum_{g_{k+d-1},...,g_{k+1},g_k}^{n,...,g_{k+2},g_{k+1}}G\sum_P\prod_{w=k}^{k+d}\sum_{t=g_{w-1}}^{g_w}(-1)^{g_w-t}\sum_{\substack{n_\tau=t-g_{w-1}\\ n_{\tau'}=g_w-t}}e^{-(\mathrm{A}+\mathrm{B})}$$

where $G=\left(\prod_{w=k}^{k+d}(g_w-g_{w-1})!\right)^{-1}$, $g_{k-1}=0$, $g_{k+d}=n$, $\sum_P$ denotes the summation over all $n!$ permutations $\{j_1,j_2,...,j_n\}$ of $\{1,2,\ldots,n\}$. $\sum_{\substack{n_\tau=t-g_{w-1}\\ n_{\tau'}=g_w-t}}$ denotes the sum over all $\binom{g_w-g_{w-1}}{t-g_{w-1}}$ subsets $\tau=\{\tau_1,\tau_2,...,\tau_{t-g_{w-1}}\}$ and $\tau'=\{\tau'_1,\tau'_2,...,\tau'_{g_w-t}\}$ of $\{j_{g_{w-1}+1},j_{g_{w-1}+2},...,j_{g_w}\}$, $\mathrm{A}=\sum_{i=1}^{t-g_{w-1}}\alpha_{\tau_i}x_w^\beta$ and $\mathrm{B}=\sum_{i=1}^{g_w-t}\alpha_{\tau'_i}x_{w-1}^\beta$.

Proof. Let us consider,

$$F_{r_k,r_{k+1},\dots,r_{k+d-1}:n}(x_k, x_{k+1},\dots,x_{k+d-1}) = \Pr(X_{r_k:n} \le x_k, X_{r_{k+1}:n} \le x_{k+1},\dots, X_{r_{k+d-1}:n} \le x_{k+d-1})$$

$$= \sum_{g_{k+d-1}=r_{k+d-1}}^{n} \dots \sum_{g_{k+1}=r_{k+1}}^{g_{k+2}} \sum_{g_k=r_k}^{g_{k+1}} \Pr(\text{exactly } g_k \text{ of } X\text{'s are at most } x_k, \text{exactly } g_{k+1} \text{ of } X\text{'s are at most } x_{k+1},\dots, \text{exactly } g_{k+d-1} \text{ of } X\text{'s are at most } x_{k+d-1})$$

$$= \sum_{g_{k+d-1}=r_{k+d-1}}^{n} \dots \sum_{g_{k+1}=r_{k+1}}^{g_{k+2}} \sum_{g_k=r_k}^{g_{k+1}} \frac{1}{(g_k)!(g_{k+1}-g_k)!\dots(g_{k+d}-g_{k+d-1})!} \sum_P \prod_{i=k}^{g_k} F_{j_i}(x_k) \prod_{i=g_k+1}^{g_{k+1}} [F_{j_i}(x_{k+1}) - F_{j_i}(x_k)] \prod_{i=g_{k+d-1}+1}^{n} [1 - F_{j_i}(x_{k+d-1})]$$

$$= \sum_{g_{k+d-1},\dots,g_{k+1},g_k}^{n,\dots,g_{k+2},g_{k+1}} \left(\prod_{w=k}^{k+d} (g_w - g_{w-1})! \right)^{-1} \sum_P \prod_{w=k}^{k+d} \prod_{i=g_{w-1}+1}^{g_w} \left[F_{j_i}(x_w) - F_{j_i}(x_{w-1})\right] \tag{7}$$

where $F_{j_i}(x_{k-1}) = 0$ and $F_{j_i}(x_{k+d}) = 1$.

Now, consider (2) it follows that

$$\prod_{i=g_{w-1}+1}^{g_w} \left[e^{-\alpha_{j_i} x_{w-1}^{\beta}} - e^{-\alpha_{j_i} x_w^{\beta}} \right] = \sum_{t=g_{w-1}}^{g_w} (-1)^{g_w - t} \sum_{\substack{n_\tau = t - g_{w-1} \\ n_{\tau'} = g_w - t}} e^{-\left(\sum_{i=1}^{t-g_{w-1}} \alpha_{\tau_i} x_w^{\beta} + \sum_{i=1}^{g_w - t} \alpha_{\tau_i'} x_{w-1}^{\beta} \right)}.$$

If above relation is used in the (7), we get the proof.

In the following theorem we will give the df of $X_{r_k:n}$ ($1 \le r_k \le n$ and $k = 1,2,\dots,n$).

Theorem 3.2.

$$F_{r_k:n}(x) = \sum_{g_k=r_k}^{n} \frac{1}{(g_k)!(n-g_k)!} \sum_P \sum_{t=1}^{g_k+1} (-1)^{g_k - t + 1} \sum_{\substack{n_\tau = t-1 \\ n_{\tau'} = g_k - t + 1}} e^{-(\mathrm{C}+\mathrm{D})} \tag{8}$$

where, $\sum_{\substack{n_\tau = t-1 \\ n_{\tau'} = g_k - t + 1}}$ denotes the sum over all $\binom{g_k}{t-1}$ subsets $\tau = \{\tau_1, \tau_2, \dots, \tau_{t-1}\}$ and $\tau' = \{\tau_1', \tau_2', \dots, \tau_{g_k - t + 1}'\}$ of $\{j_1, j_2, \dots, j_{g_k}\}$, $\mathrm{C} = \sum_{i=1}^{g_k - t + 1} \alpha_{\tau_i} x^{\beta}$ and $\mathrm{D} = \sum_{i=g_k+1}^{n} \alpha_{j_i} x^{\beta}$.

Proof. In (2), if $d = 1$, we can obtain as

$$F_{r_k:n}(x) = \sum_{g_k=r_k}^{n} \frac{1}{(g_k)!(n-g_k)!} \sum_P \left[\prod_{i=1}^{g_k} F_{j_i}(x) \right] \prod_{i=g_k+1}^{n} \left[1 - F_{j_i}(x)\right].$$

Now, consider (2) it follows that

$$F_{r_k:n}(x)=\sum_{g_k=r_k}^{n}\frac{1}{(g_k)!(n-g_k)!}\sum_{P}\prod_{i=1}^{g_k}\left[1-e^{-\alpha_{j_i}x^{\beta}}\right]e^{-\sum_{i=g_k+1}^{n}\alpha_{j_i}x^{\beta}} .$$

Upon using

$$\prod_{i=1}^{g_k}\left[1-e^{-\alpha_{j_i}x^{\beta}}\right]=\sum_{t=1}^{g_k+1}(-1)^{g_k-t+1}\sum_{\substack{n_{\tau}=t-1\\ n_{\tau'}=g_k-t+1}}\prod_{i=1}^{g_k-t+1}e^{-\alpha_{\tau_i}x^{\beta}}$$

we get the proof.

Putting $r_k=1$ and $r_k=n$ in (3), respectively, we obtain the df's of minimum and maximum order statistics from non-identically distributed continuous weibull variables.

References

[1] Arnold, B. C., Balakrishnan, N. & Nagaraja, H. N., (1992), 'A first course in order statistics', John Wiley and Sons Inc., New York.
[2] Balasubramanian, K. & Beg, M. I., (2003), 'On special linear identities for order statistics', Statistics 37: 335-339.
[3] Reiss, R. -D., (1989), 'Approximate distributions of order statistics', Springer-Verlag, New York.
[4] David, H. A., (1981), 'Order statistics', John Wiley and Sons Inc., New York.
[5] Balakrishnan, N., (1988), 'Recurrence relations for order statistics from n independent and non-identically distributed random variables', Ann. Inst. Statist. Math, 273-277.
[6] Vaughan, R. J. and Venables, W. N., (1972), 'Permanent expressions for order statistics densities', Journal of the Royal Statistical Society, Ser. B, (34): 308-310.
[7] Balasubramanian, K., Beg, M.I. & Bapat, R.B., (1991) 'On families of distributions closed under extrema', Sankhyâ, Ser.A, (53): 375-388.
[8] Balakrishnan, N., (2007), 'Permanents, order statistics, outliers and robustness', Rev. Mat. Complut., (20): 7-107.
[9] Güngör, M., (2012), 'On joint distributions of order statistics from innid variables', Bull. Malays. Math. Sci. Soc. 35: 215-225.
[10] Güngör, M., (2012), 'Identities leading to joint distributions of order statistics of innid variables from a truncated distribution', Kuwait J. Sci. Eng. 39: 1-19.
[11] Gökdere, G., (2012), ' Probability density function of order statistics of nonidentically distributed truncated continuous random variables', JESTECH, 15(4): 193-200.
[12] Henrick, J. Malik and Roger Trudel, (1982), 'Probability density function of quotient of order statistics from the pareto, power and weibull distributions', Communications in Statistics- Theory and Methods, 11: 801-814.
[13] Waleed Mohamed Afify, (2010), ' Order statistics from non-identical random variables',VDM Verlag Dr. Müller, Saarbrücken.